

Lecture 9 (Feb 29, 2016)

Region of attraction:

In case of asymptotic stability of $x=0$, for what set of initial conditions, does the trajectory converge?

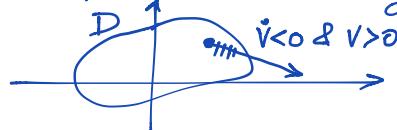
Def. Let $\phi(t, x)$ be the solution of $\dot{x} = f(x)$ which starts at initial state x at time $t=0$.

Def. (region of attraction) R.A

$$\left\{ x \in \mathbb{R}^n \mid \phi(t, x) \text{ defined } \forall t \geq 0, \text{ and } \lim_{t \rightarrow \infty} \phi(t, x) = 0 \right\}$$

Important: Neither D nor $B_r \subset D$ from proof is necessary an estimate of region of attraction.

(even though $V < 0$ & $V > 0$ in D !)



1. Not easy to compute the region of attraction. However, can get estimates.
2. In case conditions for asymptotic stability are satisfied, then if $\Omega_c = \{x \in \mathbb{R}^n \mid V(x) \leq c\}$ is bounded and contained in D , then Ω_c is a (possibly conservative) estimate of R.A.
3. A "best" estimate of R.A. is Ω_c where c is maximum value s.t. $\Omega_c \subset D$ is bounded.

Example. simple Pendulum $\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a \sin x_1 - bx_2\end{aligned}$

Recall $V = \frac{1}{4} (bx_1 + x_2)^2 + \frac{1}{4} x_2^2 + a(1 - \cos x_1)$ used to prove $(0,0)$ is a.s. where $b > 0$.

If $x = (\pi, 0)$, then $V = \frac{1}{4} (b\pi)^2 + 2a$

Let $c < \frac{1}{4} (b\pi)^2 + 2a$, then $\Omega_c = \{x \mid V(x) \leq c\}$ is an estimate of RA.

Def. If region of attraction is \mathbb{R}^n , then the origin is "globally asymptotically stable (g.a.s.)

To ensure this, need more than conditions for a.s. to hold globally.

Difficulty is in ensuring that any point $x \in \mathbb{R}^n$ is in the interior of a bounded set $\Omega_c = \{x \in \mathbb{R}^n \mid V(x) \leq c\}$.

For Ω_c to be contained inside a ball B_r , c must satisfy $c < \inf_{\|x\| \geq r} V(x)$.
If $\ell = \liminf_{r \rightarrow \infty} V(x) < \infty$ and $c < \ell$, then Ω_c : bounded.

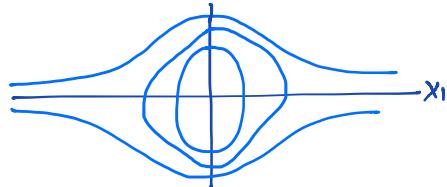
To ensure that Ω_c is bounded for all values of $c > 0$, we need $V(x)$ to be "radially unbounded", i.e., $V(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$.

e.g. $V(x) = \frac{1}{2} \|x\|^2$ is radially unbounded.

e.g. $V(x) = \frac{x_1^2}{1+x_1^2} + x_2^2$ is not radially unbounded.

small $c \rightarrow V(x) = c$: closed \rightarrow bounded

large $c \rightarrow V(x) = c$: open \rightarrow unbounded



Theorem 4.2.

Let $x=0$ be an eq. pt. of $\dot{x}=f(x)$. Let $V: \mathbb{R}^n \rightarrow \mathbb{R}$ be a C^1 & positive definite function on \mathbb{R}^n st. $\dot{V}(x)$ is negative definite. If $V(x)$ is also radially unbounded, then $x=0$ is globally asymptotically stable.

Proof. Given any point $p \in \mathbb{R}^n$, let $c = V(p)$. Radially unboundedness implies that for any $c > 0$, there is $r > 0$ st. $V(x) > c$ whenever $\|x\| > r$. Thus $\Omega_c \subset B_r$, i.e. Ω_c : bounded. Rest as before.

Note. Need origin to be the unique equilibrium!

Pendulum example

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -a \sin x_1 - bx_2$$

what about eq. pt $(\pi, 0)$?

$$V(x) = \frac{1}{2} x_2^2 + a(1 - \cos x_1)$$

$$V(\pi, 0) = 2a$$

If we start nearby for some x_1 & $x_2 = 0$, then $V(x(0)) < 2a$.

Since V decreases, we will always move away from $(\pi, 0)$.

\therefore unstable

Formalize the idea:

Theorem 4.3 Chetaev's Theorem (instability)

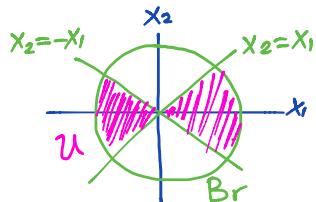
Let $x=0$ be an eq. pt. of $\dot{x}=f(x)$. Let $V: D \rightarrow \mathbb{R}$ be a C^1 function.

(i) Suppose there always exists point x_0 st. $V(x_0) > 0$ for $\|x_0\|$ arbitrary small. Suppose $V=0$.
 r: small, $B_r \subset D$

(ii) Suppose we can find a set $U = \{x \in B_r(0) \mid V(x) > 0\}$ st. $\dot{V} > 0$ on U , and the boundary of U made up of the surface $V(x)=0$ (includes origin) & $\|x\|=r$.

Then $x=0$ is unstable.

Example. The set U for $V(x) = x_1^2 - x_2^2$



Proof. For $x(t_0) \in U$, no matter how close $x(t_0)$ is to 0, $x(t)$ will escape U since $\dot{V}(x(t)) > 0$ & V is bounded in U . But $x(t)$ must escape $B_r(0)$, since $x(t)$ cannot intersect boundary surface where $V(x)=0$ ($< V(x_0)$).

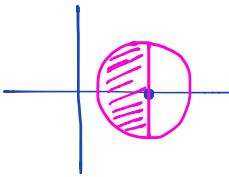
Back to pendulum:

$$\text{choose } V = a(1 + \cos x_1) - \frac{1}{4}x_2^2 - \frac{1}{4}(bx_1 + x_2)^2 + \frac{1}{4}b^2\pi^2$$

$$V(\pi, 0) = 0$$

$$\begin{aligned}\dot{V} &= -a \sin x_1 \cdot x_2 - \frac{1}{2}x_2(-bx_2 - a \sin x_1) - \frac{1}{2}(bx_2 - bx_2 - a \sin x_1)(bx_1 + x_2) \\ &= \frac{1}{2}bx_2^2 + \frac{1}{2}abx_1 \sin x_1 > 0 \text{ when } 0 < |x_1| < \pi\end{aligned}$$

$$\text{Let } \mathcal{U} = \{x \in \text{Br}(\pi, 0) \mid |x_1| < \pi\}$$



Then V & \dot{V} are positive in \mathcal{U} .

Summary:

$V: D \rightarrow \mathbb{R}$, C , Positive definite, $x^* = 0$ eq. pt of $\dot{x} = f(x)$

1) $\dot{V} \leq 0 \rightarrow 0$: stable

2) $\dot{V} < 0$ on $D - \{0\} \rightarrow 0$: a.s.

3) $\dot{V} < 0$ on $D - \{0\}$ & V : radially unbounded $\rightarrow 0$: g.a.s

LaSalle

1) same

2) $\dot{V} \leq 0$ & no solution except $x=0$ can stay in $S = \{x : \dot{V}(x)=0\}$
 $\rightarrow 0$: a.s. (above $S = \{0\}$)

3) $\dot{V} \leq 0$ & no solution except $x=0$ can stay in $S = \{x : \dot{V}(x)=0\}$
& V : radially unbounded $\rightarrow 0$: g.a.s. (above $S = \{0\}$)